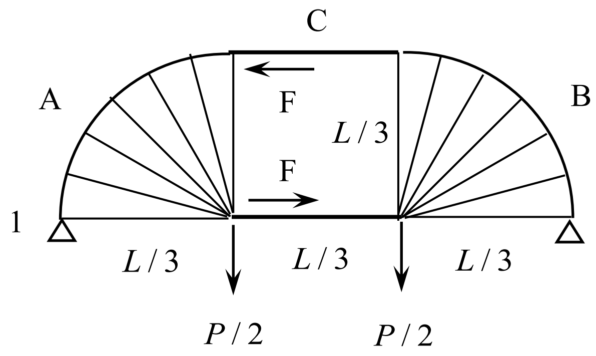
A. Rothwell

Optimization Methods in Structural Design (Springer, 2017)

Worked solutions are provided for exercises in the book having specific numerical answers (other than those obtained by use of the spreadsheets).

Exercise 1.2



Suppose we choose a truss based on a known Michell structure with span $2L/3$ and a single load P at mid-span.

For parts A and B together, from equation (1.12):

$$\text{volume } = \frac{\pi}{2} \cdot \frac{P}{\sigma_0} \cdot \frac{2L}{3} = \frac{\pi}{3} \cdot \frac{PL}{\sigma_0}$$

For part C (moments about point 1):

$$F \cdot \frac{L}{3} = \frac{P}{2} \cdot \frac{L}{3} \quad => \quad F = \frac{P}{2}$$

$$\text{cross-sectional area } = \frac{F}{\sigma_0} = \frac{P}{2\sigma_0}$$

$$\text{volume } = 2 \cdot \frac{P}{2\sigma_0} . \frac{L}{3} = \frac{1}{3} \cdot \frac{PL}{\sigma_0}$$

$$\text{Total volume } = \frac{\pi + 1}{3} \cdot \frac{PL}{\sigma_0} = \mathbf{1.381} \ \frac{PL}{\sigma_0}$$

N.B. The resulting truss is not itself a Michell structure.

The volume of the truss can be reduced if it is permitted to extend it *below* the line of the loads.

Exercise 1.3

Equation (1.15):

$$V_{\min} = \left(\frac{1}{2} + \frac{1}{6} \cdot \frac{L}{H}\right) \cdot \frac{PL}{\sigma_0}$$

$$\text{or} \quad n = \left(\frac{1}{2} + \frac{1}{6} \cdot \frac{L}{H}\right)$$

Equation (1.10):

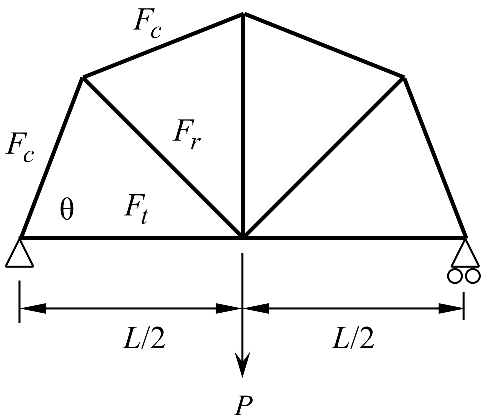
$$L_{\max} = \frac{1}{n}\left(\frac{\sigma_0}{\rho_w}\right)$$

Replace L by $L_{\max}$ in the equation for n and substitute for n in the equation for $L_{\max}$.

The resulting equation can be solved for $L_{\max}$ (for convenience use Goal Seek in Excel) to give:

$$L_{\max} = \mathbf{277.7} \text{ m}$$

Exercise 1.4



By equilibrium at the nodes: all forces F_c are equal.

All angles $\theta = 67.5°$

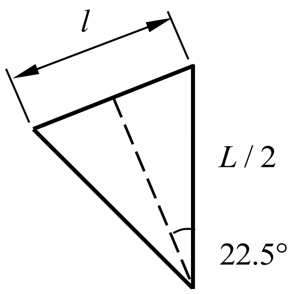
Forces in the bars:

$$F_c \sin\theta = -\frac{P}{2}, \qquad F_c = -0.5412P$$

$$F_r = 2F_c \cos\theta, \qquad F_r = 0.4142P$$

$$F_t = F_c \cos\theta, \qquad F_t = 0.2071P$$

Length of the 'circumferential' bars:



$$l = 2 \times \frac{L}{2} \times \sin(22.5°) = 0.3827L$$

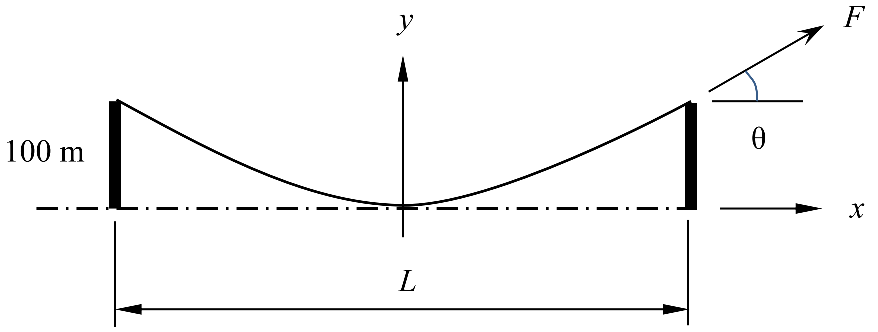
Length of all other bars = $0.5L$

Minimum volume of the truss:

$$V_{\min} = \left(4 \times 0.5412 \times 0.3827 + 3 \times 0.4142 \times 0.50 + 2 \times 0.2071 \times 0.50\right) \frac{PL}{\sigma_0}$$

$$V_{\min} = \mathbf{1.657} \frac{PL}{\sigma_0}$$

Exercise 1.5



units N, m

Formula for a parabola:

$$y = ax^2, \qquad y = 100 \ \text{at} \ x = \frac{L}{2} \quad => \quad a = \frac{400}{L^2}$$

Angle θ:

$$\theta = \left[\frac{dy}{dx}\right]_{x=L/2} = \left[2ax\right]_{x=L/2} \quad => \quad \theta = \frac{400}{L} \ \text{rad}$$

Weight of the cable with cross-sectional area A:

$$W = 7850 \times AL \times 9.807 = 76\,980AL$$

N.B. curve of the cable is assumed shallow, curved length $\approx L$

Force in the cable at each end:

$$F \sin\theta = \frac{W}{2} = 38\,490AL$$

$$=> \ F = 38\,490AL \times \frac{L}{400} = 96.23AL^2$$

N.B. θ assumed small, $\sin\theta \approx \theta$

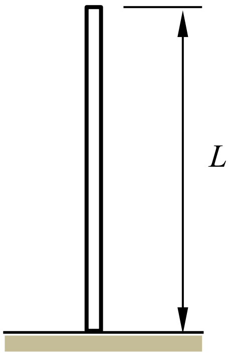
Tensile strength of the cable:

$$96.23AL_{\max}^2 = 10^9 \times A$$

$$=> \ L_{\max} = \mathbf{3224} \ \text{m}$$

Note: the curve is actually a catenary, not a parabola, but the difference is negligible when the distance between the towers is much greater than their height.

Exercise 1.6



radius $R = 0.5$ m

thickness t

density ρ

Weight of tube $W = 2\pi R t L \times 9.807\rho$

Second moment of area $I = \pi R^3 t \quad (t \ll R)$

Euler's formula:

$$P = \frac{\pi^2 E \times \pi R^3 t}{(1.122L)^2} = W \quad \text{for maximum length of tube}$$

Substituting for W gives:

$$L_{\max}^3 = \frac{\pi^2 E R^2}{2 \times (1.122)^2 \times 9.807\rho}$$

Maximum length $L_{\max} =$ **136.5** m

N.B. For a thin tube the maximum length is independent of its thickness, therefore the thickness can be increased as necessary to avoid local buckling.

Exercise 2.1

From Table 2.1, the efficiency formula for a square section tube:

$$\sigma_{max} = 0.907 E^{3/5} \left(\frac{P}{L^2} \right)^{2/5}$$

At the limiting value of P / L^2:

$$\sigma_{max} = 300 \text{ N/mm}^2 \quad \text{(allowable stress of the material)}$$

Therefore:

$$\left(\frac{P}{L^2} \right)^{2/5} = \frac{300}{0.907 \times (72\,000)^{3/5}} = 0.4028$$

$$\Rightarrow \quad \frac{P}{L^2} = \mathbf{0.1030} \text{ N/mm}^2$$

Exercise 2.2

Hexagonal tube:

cross sectional area $A = 6bt$, $I = \frac{5}{2}b^3t$

Euler buckling stress:

$$\sigma_E = \frac{\pi^2 EI}{L^2} \cdot \frac{1}{6bt} = \frac{5\pi^2}{12}E\left(\frac{b}{L}\right)^2 = K_F E\left(\frac{b}{L}\right)^2 \text{, say}$$

Local buckling stress:

$$\sigma_L = 3.62E\left(\frac{t}{b}\right)^2 = K_L E\left(\frac{t}{b}\right)^2 \text{, say}$$

Compressive stress:

$$\sigma = \frac{P}{6bt} = \frac{P}{Cbt} \text{ , say}$$

Simultaneous modes of buckling:

$$\sigma = \frac{P}{Cbt} = K_F E\left(\frac{b}{L}\right)^2 = K_L E\left(\frac{t}{b}\right)^2$$

Eliminating b and t from the above three equations leaves (after some algebra):

$$\sigma = \left(\frac{K_F{}^2 K_L}{C^2}\right)^{1/5} E^{3/5}\left(\frac{P}{L^2}\right)^{2/5}$$

with $t = \left(\frac{K_F}{K_L}\right)^{1/2}\frac{b^2}{L}$

$$b = \left(\frac{K_L}{C^2 K_F{}^3}\right)^{1/10}\left(\frac{PL^3}{E}\right)^{1/5}$$

From the formula for σ above:

efficiency $\eta = \left(\frac{K_F{}^2 K_L}{C^2}\right)^{1/5}$

Substituting:

$$K_F = \frac{5\pi^2}{12} \text{ , } K_L = 3.62 \text{ , } C = 6$$

gives: $\eta = \mathbf{1.112}$

Exercise 2.6

$$P = 10\,000 \text{ N}, \quad L = 1000 \text{ mm}, \quad E = 72\,000 \text{ N/mm}^2$$

minimum thickness (mm)	efficiency	
0.191	0.780	optimum
0.2	0.757	
0.4	0.477	
0.6	0.364	
0.8	0.301	
1.0	0.259	
1.2	0.229	
1.4	0.207	
1.6	0.189	
1.8	0.175	
2.0	0.163	

Exercise 3.1

Lagrangian function:

$$F(\mathbf{x},\lambda) = x_1{}^2 + x_2{}^2 + x_3{}^2 - \lambda(x_1 + 2x_2 + 3x_3 - 7)$$

Differentiating:

$$\frac{\partial F}{\partial x_1} = 2x_1 - \lambda = 0 \qquad \Rightarrow \qquad \lambda = 2x_1$$

$$\frac{\partial F}{\partial x_2} = 2x_2 - 2\lambda = 0 \qquad \Rightarrow \qquad x_2 = 2x_1$$

$$\frac{\partial F}{\partial x_3} = 2x_3 - 3\lambda = 0 \qquad \Rightarrow \qquad x_3 = 3x_1$$

$$\frac{\partial F}{\partial \lambda} = x_1 + 2x_2 + 3x_3 - 7 = 0$$

Substituting in the last condition:

$$x_1 + 4x_1 + 9x_1 - 7 = 0$$

$$\Rightarrow \quad x_1 = \frac{1}{2}$$

$$x_2 = 1$$

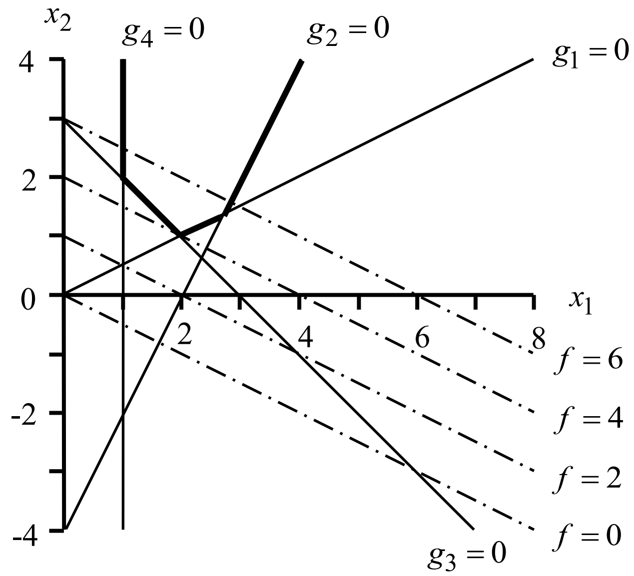
$$x_3 = \frac{3}{2}$$

$$\lambda = 1$$

Substituting in $f(\mathbf{x})$:

$$f_{\min} = \left(\frac{1}{2}\right)^2 + 1 + \left(\frac{3}{2}\right)^2 = \mathbf{\frac{7}{2}}$$

Exercise 3.2



From the design space, the active constraints are $g_1(\mathbf{x})$ and $g_3(\mathbf{x})$.

Lagrangian function (now treating $g_1(\mathbf{x})$ and $g_3(\mathbf{x})$ as equalities):

$$F(\mathbf{x}, \boldsymbol{\lambda}) = 2x_2 + x_1 - \lambda_1(2x_2 - x_1) - \lambda_2(x_2 + x_1 - 3)$$

Differentiating:

$$\frac{\partial F}{\partial x_1} = 1 + \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial F}{\partial x_2} = 2 - 2\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial F}{\partial \lambda_1} = 2x_2 - x_1 = 0$$

$$\frac{\partial F}{\partial \lambda_2} = x_2 + x_1 - 3 = 0$$

Solving the above equations:

$$x_1 = 2, \qquad x_2 = 1, \qquad \lambda_1 = \frac{1}{3}, \qquad \lambda_3 = \frac{4}{3}$$

Substituting in $f(\mathbf{x})$:

$$f_{\min} = \mathbf{4}$$

Exercise 3.3

Total surface area:

$$A = 2\pi R^2 + 2\pi RL$$

Internal volume:

$$V = \pi R^2 L$$

Lagrangian function to minimize A subject to $V = V_0$:

$$F(R, L, \lambda) = 2\pi R^2 + 2\pi RL - \lambda(\pi R^2 L - V_0)$$

Differentiating:

$$\frac{\partial F}{\partial R} = 4\pi R + 2\pi L - 2\pi RL\lambda = 0$$

$$\frac{\partial F}{\partial L} = 2\pi R - \pi R^2 \lambda = 0$$

$$\frac{\partial F}{\partial \lambda} = -\pi R^2 L + V_0 = 0$$

Solving the above equations for R, L, and λ:

$$R = \left(\frac{V_0}{2\pi}\right)^{1/3}$$

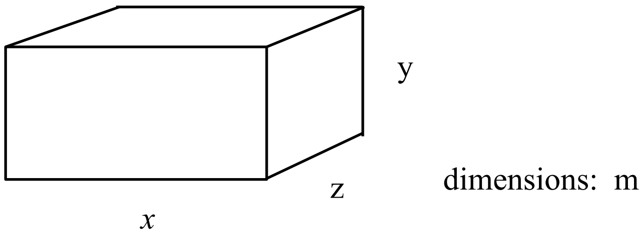
$$L = \left(\frac{4V_0}{\pi}\right)^{1/3}$$

$$\lambda = \left(\frac{16\pi}{V_0}\right)^{1/3}$$

Substituting in the expression for A, the minimum surface area:

$$A_{\min} = 2\pi \left(\frac{V_0}{2\pi}\right)^{2/3} + 2\pi \left(\frac{V_0}{2\pi}\right)^{1/3} \left(\frac{4V_0}{\pi}\right)^{1/3} \;=\; \mathbf{5.536}\ V_0^{2/3}$$

Exercise 3.4



N.B. Because all four sides have the same cost, we can assume $z = x$, leaving only two variables x and y.

Therefore volume of the container $V = x^2 y$

Cost $= x^2 \times 20 + 4xy \times 30 + x^2 \times 10 = 30x^2 + 120xy$

Lagrangian function to minimize the cost subject to $V = 4$ m^3:

$$F(x, y) = 30x^2 + 120xy - \lambda(x^2 y - 4)$$

Differentiating:

$$\frac{\partial F}{\partial x} = 60x + 120y - 2\lambda xy = 0$$

$$\frac{\partial F}{\partial y} = 120x - \lambda x^2 = 0$$

$$\frac{\partial F}{\partial \lambda} = x^2 y - 4 = 0$$

Solving the above equations for x and y:

$x = z = 2$ m

$y = 1$ m

$\lambda = 60$ €/m^3 (note that λ is positive for <u>minimum</u> cost)

Minimum cost $= 30 \times (2)^2 + 120 \times 2 \times 1 =$ € **360**

Lagrange multiplier λ is the rate of change of cost with volume.

Therefore the estimated minimum cost if the volume is increased to 5 m^3

$= 360 + 60 \times (5 - 4) =$ € **420**

Exercise 3.5

Variables are the radius R and thickness t.

Minimize the cross-sectional area $A = 2\pi Rt$

subject to constraints:

$$\frac{\pi^3 ER^3 t}{L^2} - P \geq 0 \quad \text{(Euler buckling)}$$

$$2\pi KEt^2 - P \geq 0 \quad \text{(local buckling)}$$

Lagrangian function:

$$F(R,t) = 2\pi Rt - \lambda_1 \left(\frac{\pi^3 ER^3 t}{L^2} - P \right) - \lambda_2 \left(2\pi KEt^2 - P \right)$$

Differentiating:

$$\frac{\partial F}{\partial R} = 2\pi t - \lambda_1 \left(\frac{3\pi^3 ER^2 t}{L^2} \right) = 0$$

$$\frac{\partial F}{\partial t} = 2\pi R - \lambda_1 \left(\frac{\pi^3 ER^3}{L^2} \right) - \lambda_2 \left(4\pi KEt \right) = 0$$

$$\frac{\partial F}{\partial \lambda_1} = \frac{\pi^3 ER^3 t}{L^2} - P = 0$$

$$\frac{\partial F}{\partial \lambda_2} = 2\pi KEt^2 - P = 0$$

Solving the above equations for R, t, λ_1, λ_2:

$$t = \left(\frac{P}{2\pi KE} \right)^{1/2},$$

$$R = \left(\frac{PL^2}{\pi^3 Et} \right)^{1/3} = \left(\frac{2KPL^4}{\pi^5 E} \right)^{1/6}$$

$$\lambda_1 = \frac{2L^2}{3\pi^2 ER^2}$$

$$\lambda_2 = \frac{R}{3KEt} = \frac{4\pi R}{3}$$

Note that Lagrange multipliers λ_1 and λ_2 are positive, non-zero (both constraints active) for all values of P, L and E, implying that the optimum R and t invariably correspond to simultaneous buckling in Euler and local buckling.

Exercise 3.8

eccentricity (mm)	cross-sectional area (mm^2)	diameter (mm)	thickness (mm)
0	44.4	61.2	0.232
10	74.1	89.8	0.264
20	91.4	102.0	0.286
30	106.1	111.1	0.305
40	119.4	118.7	0.321
50	131.6	125.3	0.335

Exercise 3.9

compressive load (N)	cross-sectional area (mm^2)	compressive stress (N/mm^2)
5000	23.9	209
10 000	44.4	225
20 000	84.6	236
40 000	163.0	245

Exercise 4.5

$$f(x) = 2x^3 - 3x + 2$$

x_L	x_U	x_1	x_2	f_1	f_2	$x_U - x_L$	action
0	1	0.3820	0.6180	0.9656	0.6180	1.0000	replace x_L by x_1
0.3820	1.0000	0.6180	0.7639	0.6180	0.5999	0.6180	replace x_L by x_1
0.6180	1.0000	0.7639	0.8541	0.5999	0.6838	0.3820	replace x_U by x_2
0.6180	0.8541	0.7082	0.7639	0.5858	0.5999	0.2361	replace x_U by x_2
0.6180	0.7639	0.6738	0.7082	0.5904	0.5858	0.1459	replace x_L by x_1
0.6738	0.7639	0.7082	0.7295	0.5858	0.5879	0.0902	replace x_U by x_2
0.6738	0.7295						

$$0.6738 \le x \le 0.7295$$
$$f_{\min} \le 0.5858$$

A more accurate result can be obtained by continuing the table.

Exercise 4.6

Based on the table made in Exercise 4.5, select points:

$$x_1 = 0.65, \qquad x_2 = 0.7, \qquad x_3 = 0.75$$

$$\text{with:} \quad f_1 = 0.5993 \quad f_2 = 0.5860 \quad f_3 = 0.5938$$

Substituting in equation (4.8) for parabolic interpolation:

$$x = \mathbf{0.7065}$$

$$f_{\min} = \mathbf{0.5858}$$

By differentiation:

$$f = 2x^3 - 3x + 2$$

$$\frac{df}{dx} = 6x^2 - 3 = 0 \quad \text{for a minimum}$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{2}} = 0.7071$$

$$f_{\min} = 0.5858$$

Exercise 5.1

Intersection line passes through the points:

$$(2, 0, 0) \text{ and } (0, 2, 2)$$

Minimum is at:

$$x_1 = \frac{4}{3}, \quad x_2 = x_3 = \frac{2}{3}$$

Lagrange multipliers at this point:

$$\nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 2.6667 \\ 1.3333 \\ 1.3333 \end{bmatrix}$$

$$(\mathbf{N}^T\mathbf{N})^{-1}\mathbf{N}^T = \begin{bmatrix} 0.2222 & 0.1111 & 0.1111 \\ 0.1111 & -0.4444 & 0.5556 \end{bmatrix} \qquad \text{(previously calculated)}$$

$$\boldsymbol{\lambda} = (\mathbf{N}^T\mathbf{N})^{-1}\mathbf{N}^T\nabla f = \begin{bmatrix} 0.2222 & 0.1111 & 0.1111 \\ 0.1111 & -0.4444 & 0.5556 \end{bmatrix} \begin{bmatrix} 2.6667 \\ 1.3333 \\ 1.3333 \end{bmatrix} = \begin{bmatrix} 0.8889 \\ 0.4444 \end{bmatrix}$$

$$\lambda_1 = \mathbf{0.8889}, \quad \lambda_2 = \mathbf{0.4444} \qquad (\lambda_1, \lambda_2 > 0)$$

Analytical solution:

$$g_1(\mathbf{x}) = 3x_1 + 2x_2 + x_3 - 6 = 0$$
$$g_2(\mathbf{x}) = x_3 - x_2 = 0$$

(both constraints active at the minimum)

Solving for x_2 and x_3:

$$x_3 = x_2$$
$$x_2 = 2 - x_1$$

Substituting for x_2 and x_3 in the function:

$$f(\mathbf{x}) = 3x_1^2 - 8x_1 + 8$$

Differentiating:

$$\frac{df}{dx_1} = 6x_1 - 8 = 0 \qquad \text{for a minimum}$$

$$x_1 = \frac{4}{3}$$

$$x_2 = x_3 = \frac{2}{3}$$

Exercise 5.2

For the projection matrix **P**:

$$\mathbf{N} = \begin{bmatrix} \partial g / \partial x_1 \\ \partial g / \partial x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{N}^T \mathbf{N} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5$$

$$\left[\mathbf{N}^T \mathbf{N}\right]^{-1} = \frac{1}{5}$$

$$\left[\mathbf{N}^T \mathbf{N}\right]^{-1} \mathbf{N}^T = \frac{1}{5}\begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$\mathbf{N}\left[\mathbf{N}^T \mathbf{N}\right]^{-1} \mathbf{N}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

$$\mathbf{P} = \mathbf{I} - \mathbf{N}\left[\mathbf{N}^T \mathbf{N}\right]^{-1} \mathbf{N}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Choose an initial point: $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{4}$ (on the constraint).

At this point:

$$\nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

Search direction:

$$\mathbf{s} = -\mathbf{P}\nabla f = -\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = -\begin{bmatrix} \frac{3}{5} \\ -\frac{3}{10} \end{bmatrix}$$

Perform a line search along the line:

$$x_1 = \frac{1}{2} - 2\alpha$$

(after scaling)

$$x_2 = \frac{1}{4} + \alpha$$

from the initial point.

$$f(x) = x_1^2 + x_2^2 = \left(\frac{1}{2} - 2\alpha\right)^2 + \left(\frac{1}{4} + \alpha\right)^2$$

$$\frac{df}{dx} = -4\left(\frac{1}{2} - 2\alpha\right) + 2\left(\frac{1}{4} + \alpha\right) = 0 \quad \text{for a minimum,}$$

giving:

$$\alpha = \frac{3}{20}$$

$$x_1 = \frac{1}{5}$$

(note that this point lies on the constraint)

$$x_2 = \frac{2}{5}$$

$$f_{\min} = \mathbf{\frac{1}{5}}$$

Exercise 5.3

After the initial line search:

$$x_1 = 1.417 \qquad x_4 = 0.5$$
$$x_2 = 0.75 \qquad x_5 = 0$$
$$x_3 = 0.75$$

(note that surplus variables $x_4\ ,x_5 \geq 0$)

With linear constraints matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ are unchanged from the first iteration:

$$\mathbf{N}_1^{-1}\mathbf{N}_2 = \begin{bmatrix} 1 & -0.3333 & -0.6667 \\ -1 & 0 & 1 \end{bmatrix}$$

At the above point:

$$\nabla_u f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 2.8334 \\ 1.5 \end{bmatrix}$$

(same choice of dependent and independent variables)

$$\nabla_v f = \begin{bmatrix} 2x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \\ 0 \end{bmatrix}$$

Reduced gradient:

$$\mathbf{G}_r = \nabla_v f - \left[\mathbf{N}_1^{-1}\mathbf{N}_2\right]^T \nabla_u f = \begin{bmatrix} 1.5 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -0.3333 & 0 \\ -0.6667 & 1 \end{bmatrix} \begin{bmatrix} 2.8334 \\ 1.5 \end{bmatrix}$$

Performing the matrix calculations:

$$\mathbf{G}_r = \begin{bmatrix} 0.1666 \\ 0.9445 \\ 0.3889 \end{bmatrix}$$

New search direction in the independent variables x_3, x_4, x_5:

$$\mathbf{s} = \begin{bmatrix} -0.1666 \\ -0.9445 \\ 0 \end{bmatrix}$$

(third term set to zero to avoid x_5 becoming negative)

Dependent variables x_1, x_2:

$$d\mathbf{u} = -\mathbf{N}_1^{-1}\mathbf{N}_2 d\mathbf{v} = \begin{bmatrix} 1 & -0.3333 & -0.6667 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.1666 \\ -0.9445 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1482 \\ -0.1666 \end{bmatrix}$$

(increments in x_1, x_2 corresponding to **s**)

In terms of parameter α:

$$x_1 = 1.4167 - 0.1482\,\alpha$$

$$x_2 = 0.75 - 0.1666\,\alpha$$

$$x_3 = 0.75 - 0.1666\,\alpha$$

$$x_4 = 0.5 - 0.9445\,\alpha$$

$$x_5 = 0$$

The line search terminates when x_4 becomes zero, with no intervening minimum of the function $f(\mathbf{x})$, therefore:

$$\alpha = \frac{0.5}{0.9445} = 0.5294$$

Substituting for α in the expressions for the variables:

$$\begin{aligned} x_1 &= 1.3382 \\ x_2 &= 0.6618 \\ x_3 &= 0.6618 \\ x_4 &= x_5 = 0 \end{aligned} \qquad f(\mathbf{x}) = 2.667$$

Exercise 6.1

Second moment of area of the beam:

$$I = \frac{bh^3 - (b-t)(h-2t)^3}{12}$$

with height h = 100 mm, width b = 100 mm, both taken to be maximum for minimum n_h, and thickness t to be determined.

Set the maximum stress:

$$\sigma = \frac{Mh}{2I} = \frac{50\times 10^6 \times 100}{2I} = 500\ \text{N/mm}^2$$

Substitute for I and solve for t (use Goal Seek in Excel):

$$t = 11.61\ \text{mm}$$

Cross-sectional area:

$$A = bh - (b-t)(h-2t) = 3214\ \text{mm}^2$$

From equation (6.2):

$$n_h = \frac{A\sigma h}{M} = \frac{3214\times 500\times 100}{50\times 10^6} = \mathbf{3.214}$$

Exercise 6.2

Take an arbitrary diameter $d = 20$ mm and allowable stress $\sigma_0 = 500$ N/mm^2 (C and n_g depend only on the geometric *shape*, and are independent therefore of the values of d and σ_0)

Solid circular section:

$$A = \frac{\pi d^2}{4} = 314.2 \text{ mm}^2$$

$$I = \frac{\pi d^4}{64} = 7854 \text{ mm}^4$$

$$C = \frac{I}{A^2} = 0.07958$$

$$M = 2\frac{\sigma_0 I}{d} = 392700 \text{ Nmm}$$

$$n_g = \frac{A}{(M / \sigma_0)^{2/3}} = \mathbf{3.691}$$

Hollow circular section:

$$A = \frac{\pi d^2}{4} - \frac{\pi}{4}\left(\frac{d}{2}\right)^2 = 235.6 \text{ mm}^2$$

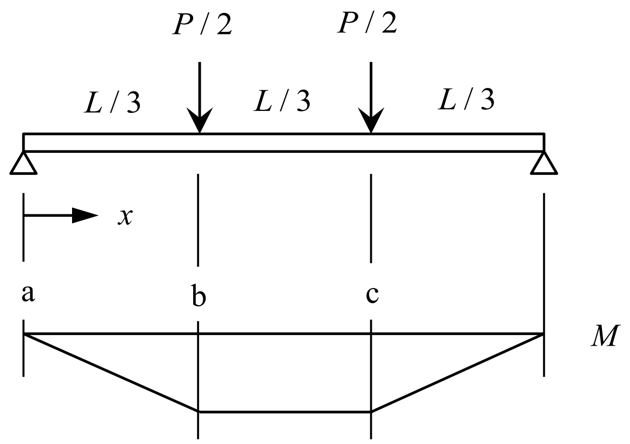
$$I = \frac{\pi d^4}{64} - \frac{\pi}{64}\left(\frac{d}{2}\right)^4 = 7363 \text{ mm}^4$$

$$C = \frac{I}{A^2} = 0.1326$$

$$M = 2\frac{\sigma_0 I}{d} = 368150 \text{ Nmm}$$

$$n_g = \frac{A}{(M / \sigma_0)^{2/3}} = \mathbf{2.890}$$

Exercise 6.3



$$M_{\max} = \frac{PL}{6}$$

Span a–b:

$$M = \frac{P}{2}x$$

$$A = n_g \left(\frac{M}{\sigma_0} \right)^{2/3} \qquad \text{(equation 6.5)}$$

$$V = \int_0^{L/3} A dx = \int_0^{l/3} n_g \left(\frac{P}{2\sigma_0} \right)^{2/3} x^{2/3} dx \; = 0.06057 n_g \left(\frac{PL}{\sigma_0} \right)^{2/3} L$$

Span b–c:

$$M = \frac{PL}{6}$$

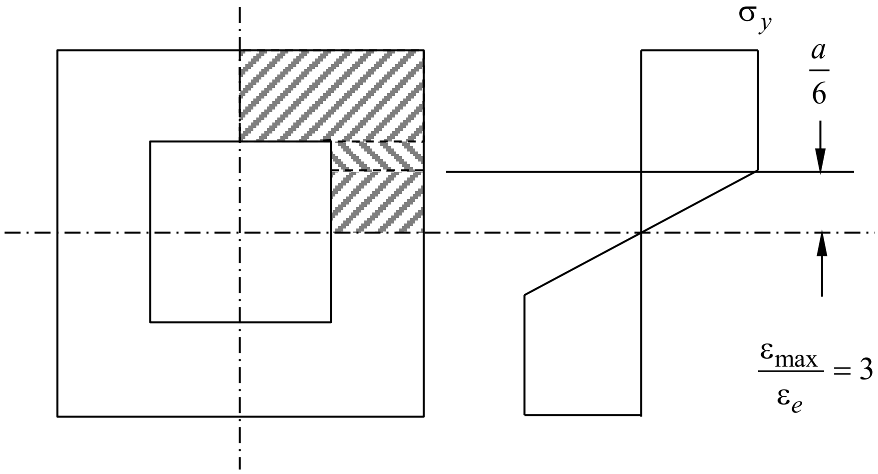
$$V = A\frac{L}{3} = n_g \left(\frac{PL}{6\sigma_0} \right)^{2/3} \frac{L}{3} = 0.10095 n_g \left(\frac{PL}{\sigma_0} \right)^{2/3} L$$

Total volume:

$$V = (2 \times 0.06057 + 0.10095)\, n_g \left(\frac{PL}{\sigma_0} \right)^{2/3} L$$

$$V = 0.2221\, n_g \left(\frac{PL}{\sigma_0} \right)^{2/3} L$$

Exercise 6.4



Elastic region:

$$\sigma = 6\frac{\sigma_y}{a}$$

Yield moment:

$$M_y = 4\int_0^{a/6} \frac{6\sigma_y}{a} y \cdot \frac{a}{4} y \cdot dy + 4\int_{a/6}^{a/4} \sigma_y \cdot \frac{a}{4} y \cdot dy + 4\int_{a/4}^{a/2} \sigma_y \cdot \frac{a}{2} y \cdot dy$$

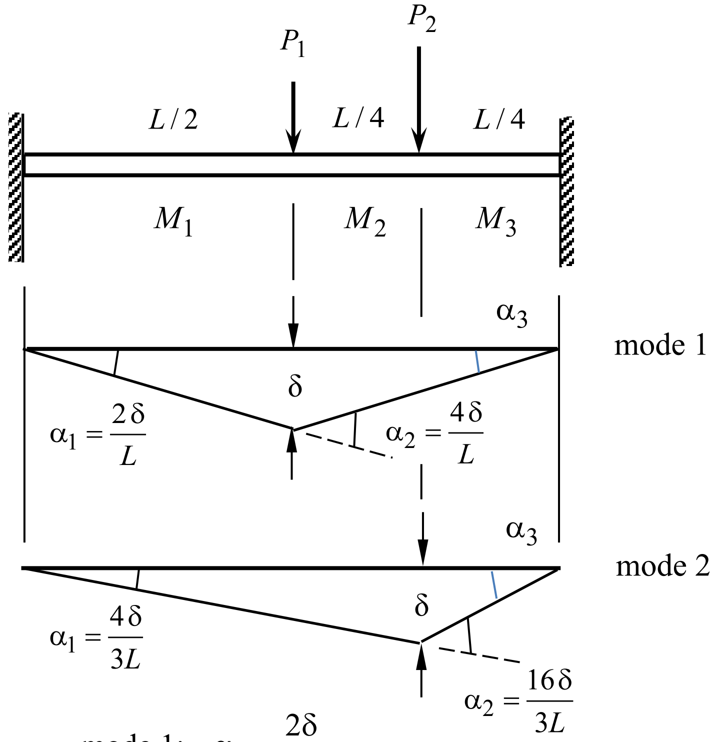
After integration:

$$M_y = 0.2141\, a^3 \sigma_y$$

Maximum elastic moment with maximum stress σ_y:

$$M_e = 0.1563\, a^3 \sigma_y$$

Exercise 6.5



mode 1: $\alpha_3 = \dfrac{2\delta}{L}$

mode 2: $\alpha_3 \dfrac{4\delta}{L}$

The formulae in Example 6.2 are modified for the additional work done at the yield hinge at the right hand end:

mode 1:
$$P_1 + \frac{P_2}{2} = 2\frac{M_1}{L} + 4\frac{M_1}{L} + 2\frac{M_3}{L} \quad (M_1 \le M_2)$$

or
$$P_1 + \frac{P_2}{2} = 2\frac{M_1}{L} + 4\frac{M_2}{L} + 2\frac{M_3}{L} \quad (M_2 \le M_1)$$

mode 2:
$$\frac{2P_1}{3} + P_2 = \frac{4}{3}\cdot\frac{M_1}{L} + \frac{16}{3}\cdot\frac{M_2}{L} + 4\frac{M_3}{L} \quad (M_2 \le M_3)$$

or
$$\frac{2P_1}{3} + P_2 = \frac{4}{3}\cdot\frac{M_1}{L} + \frac{16}{3}\cdot\frac{M_3}{L} + 4\frac{M_3}{L} \quad (M_3 \le M_2)$$

The least of the above is the correct limit load.

With loads $P_1 = 300$ N, $P_2 = 1200$ N and $L = 1000$ mm we obtain the following inequalities:

$$6M_1 + 2M_3 \geq 900 \times 10^3 \text{ Nmm}$$

$$2M_1 + 4M_2 + 2M_3 \geq 900 \times 10^3 \text{ Nmm}$$

$$\frac{4}{3}M_1 + \frac{16}{3}M_2 + 4M_3 \geq 1400 \times 10^3 \text{ Nmm}$$

$$\frac{4}{3}M_1 + \frac{28}{3}M_3 \geq 1400 \times 10^3 \text{ Nmm}$$

The volume of the beam is unchanged:

$$V = 0.0410\left(\frac{M_1^{2/3}}{2} + \frac{M_2^{2/3}}{4} + \frac{M_3^{2/3}}{4}\right)L$$

With variables M_1, M_2, M_3 and the above constraints, set up a spreadsheet and use Solver to minimize volume V and find optimum values of the yield moments:

$$M_1 = M_2 = 75 \times 10^3 \text{Nmm}$$

$$M_3 = 225 \times 10^3 \text{Nmm}$$

$$V = 92.66 \times 10^3 \text{ mm}^3$$

Both of modes 1 and the first of modes 2 are active at the optimum.

For a uniform beam clamped at both ends:

$$M_1 = M_2 = M_3 = 131.3 \times 10^3 \text{ Nmm}$$

$$V = 106.0 \times 10^3 \text{ mm}^3$$

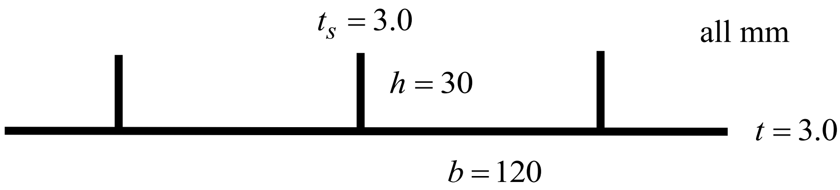
Exercise 6.6

$EI_{\min}$ (Nmm^2)	cross-sectional area (mm^2)
20×10^9	369.4
50×10^9	369.4
100×10^9	388.5
200×10^9	494.1
500×10^9	712.8

Exercise 6.7

minimum flange width (mm)	cross-sectional area (mm^2)
20	367.7
25	369.4
30	376.3
35	383.0
40	392.4
45	406.5
50	423.9

Exercise 7.1



Calculate the following:

Cross-sectional area of the panel (per stiffener) $A =$ 445.5 mm^2

Height of the neutral axis above the mid-plane of the plate = 3.023 mm

Second moment of area about the neutral axis (per stiffener) $I =$ 23 200 mm^4

Flexural buckling load (per stiffener):

$$P_F = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 72\,800 \times 23\,200}{(800)^2} = 26\,050 \text{ N}$$

Flexural buckling stress:

$$\sigma_F = \frac{P_F}{A} = \frac{26\,050}{445.5} = \mathbf{58.47} \text{ N/mm}^2$$

Local buckling stress (with $h/b = 0.25,\ t_s/t = 1.0,\ K = 3.95$):

$$\sigma_L = KE\left(\frac{t}{b}\right)^2 = 3.95 \times 72\,800 \times \left(\frac{3.0}{120}\right)^2 = \mathbf{179.7} \text{ N/mm}^2$$

The critical mode is flexural buckling.

Equivalent thickness of the panel:

$$\bar{t} = \frac{A}{b} = \frac{445.5}{120} = 3.713 \text{ mm}$$

Loading intensity:

$$p = \sigma\bar{t} = 58.47 \times 3.713 = 217.1 \text{ N/mm}$$

Efficiency:

$$\eta = \sigma / \sqrt{\frac{pE}{L}} = 58.47 / \sqrt{\frac{217.1 \times 72\,800}{800}} = \mathbf{0.416}$$

The low efficiency is due to the large difference in the flexural and local buckling stresses.

Exercise 7.2

Shear buckling coefficient, from Table 7.3:

$$\frac{h}{d} = 2, \quad \mu_c = 0.68, \quad K = 5.90$$

At shear force $Q = 120$ kN:

$$q = \frac{Q}{h} = 240 \text{ N/mm}, \quad r = \frac{d}{h} = 0.5$$

Plate thickness, from equation (7.31):

$$t = \frac{r^{2/3}}{K^{1/3}} \cdot \left(\frac{qh^2}{E}\right)^{1/3} = \frac{0.5^{2/3}}{5.90^{1/3}} \cdot \left(\frac{240 \times 500^2}{72800}\right)^{1/3} = 3.269 \text{ mm}$$

Equivalent thickness of the stiffeners, from equation (7.30):

$$t' = \frac{\mu}{C^{1/2}} \cdot h \left(\frac{t}{d}\right)^{3/2} = \frac{0.68}{1.0} \times 500 \times \left(\frac{3.269}{250}\right)^{3/2} = 0.5084 \text{ mm}$$

Total equivalent thickness:

$$\overline{t} = t + t' = 3.269 + 0.508 = 3.777 \text{ mm}$$

Equivalent shear stress:

$$\tau' = \frac{q}{\overline{t}} = \frac{240}{3.777} = \mathbf{63.54} \text{ N/mm}^2$$

Repeating the above calculation:

At shear force $Q = 60$ kN:

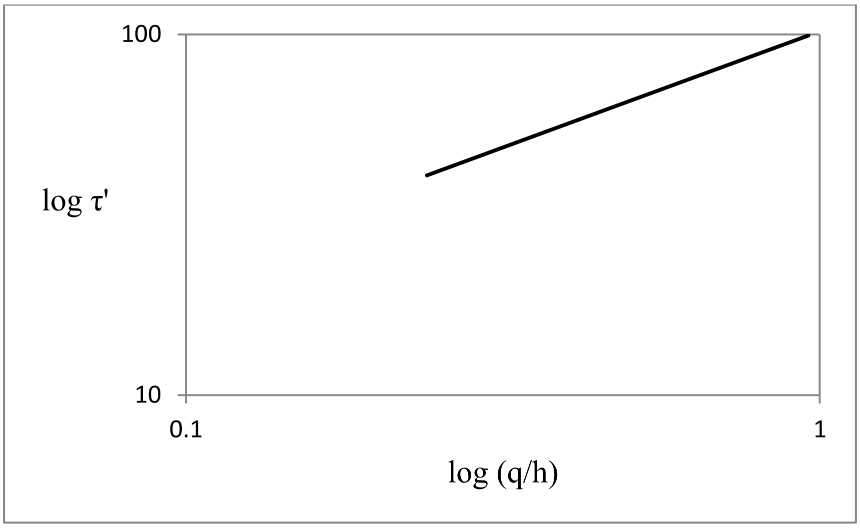
$$\tau' = \mathbf{40.64} \text{ N/mm}^2$$

At shear force $Q = 240$ kN:

$$\tau' = \mathbf{99.24} \text{ N/mm}^2$$

The $\log(\tau') - \log(q/h)$ plot below shows a linear relation,

verifying the form of efficiency formula .



Substituting in the efficiency formula for $\tau' = 240$ and 60 kN ($q/h = 0.96$ and 0.24 N/mm^2):

$$\frac{99.24}{40.64} = \left(\frac{0.96}{0.24}\right)^n \Rightarrow n = \mathbf{0.644}$$

$$99.24 = \alpha \times 72\,800 \times (0.96)^{0.644} \Rightarrow \alpha = \mathbf{1.899}$$

Note that index *n* is close to 2/3, as in the efficiency formula in section 7.3.2.

Exercise 7.5

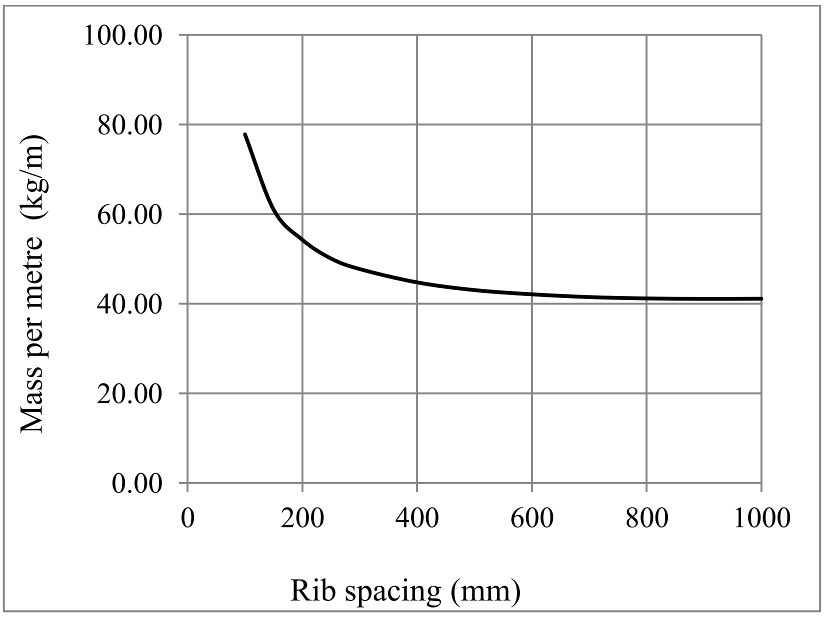
(mm)	σ_F (N/mm^2)	σ_L (N/mm^2)	$\Delta\sigma_F$ (N/mm^2)	$\Delta\sigma_L$ (N/mm^2)
b				
120	58.454	179.27		
121.2	58.073	175.76	-0.381	-3.51
h				
30	58.454	179.27		
30.3	59.998	179.02	1.544	-0.25
t				
3	58.454	179.27		
3.03	58.093	182.34	-0.361	3.07
t_s				
3	58.454	179.27		
3.03	58.835	179.83	0.381	0.56

Cross-sectional dimensions and other data as in Exercise 7.1

Exercise 7.6

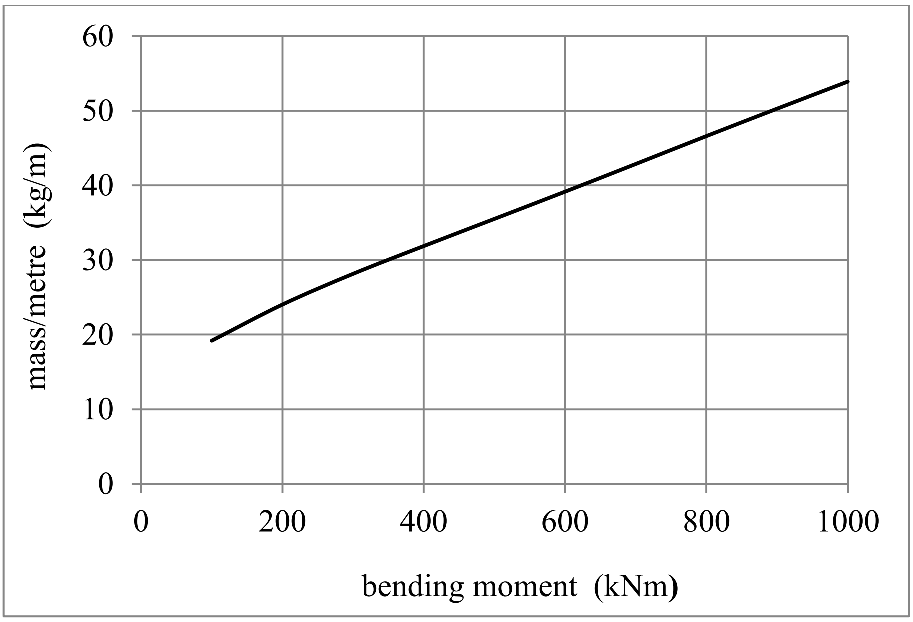
rib spacing (mm)	mass/metre (kg/m)
100	77.83
150	61.02
200	54.29
250	50.11
300	47.76
400	44.79
500	43.07
600	42.13
700	41.51
800	41.21
900	41.11
1000	41.14

Minimum mass per metre = 41.10 kg/m at *L* = 906.0 m



Exercise 7.7

M (kNm)	Q (kN)	mass/metre (kg/m)	increment
100	20	19.19	
200	40	24.03	4.84
300	60	28.15	4.12
400	80	31.87	3.72
500	100	35.51	3.64
600	120	39.16	3.65
700	140	42.87	3.71
800	160	46.61	3.74
900	180	50.28	3.67
1000	200	53.91	3.63



Exercise 7.8

frame spacing L	$t_{min} = 1.6$ mm $b_{min} = 120$ mm	$t_{min} = 0$ $b_{min} = 0$
(mm)	mass / metre (kg/m)	mass / metre (kg/m)
200	85.05	53.94
300	74.21	43.48
400	69.55	39.84
500	67.28	38.31
600	66.10	37.87
700	65.46	38.31
800	65.17	39.06
900	65.13	39.99
1000	65.24	41.00
optimum	65.13 kg/m at L = 871.5 mm	37.87 kg/m at L = 608.4 mm

Exercise 7.9

With constraints $b_{\min} = 120$ mm, $t_{\min} = 1.6$ mm :

loading case 1, location 9

$$\sigma = 118.5 \text{ N/mm}^2,\ t = 2.090 \text{ mm},\ A_s / b = 0.693 \text{ mm}$$

$$\bar{t} = t + A_s / b = 2.783 \text{ mm}$$

$$p = \sigma \bar{t} = 329.8 \text{ N/mm}$$

$$\sqrt{\frac{pE}{L}} = 219.1 \text{ N/mm}^2$$

$$\text{efficiency } \eta = \sigma / \sqrt{\frac{pE}{L}} = \mathbf{0.541}$$

After removing constraints on b and t :

loading case 1, location 9

$$\sigma = 210.1 \text{ N/mm}^2,\ t = 0.753 \text{ mm},\ A_s / b = 0.910 \text{ mm}$$

$$\bar{t} = t + A_s / b = 1.663 \text{ mm}$$

$$p = \sigma \bar{t} = 349.4 \text{ N/mm}$$

$$\sqrt{\frac{pE}{L}} = 225.5 \text{ N/mm}^2$$

$$\text{maximum efficiency } \eta_{\max} = \sigma / \sqrt{\frac{pE}{L}} = \mathbf{0.932}$$

Exercise 8.1

With 16 plies, the laminate thickness:

$$t = 16 \times 0.125 = 2.0 \text{ mm}.$$

With the same elastic constants, the **Q** matrix for all plies is unchanged from Example 8.1:

$$\mathbf{Q} = \begin{bmatrix} 150905 & 3018 & 0 \\ 3018 & 10060 & 0 \\ 0 & 0 & 6000 \end{bmatrix} \text{ N/mm}^2.$$

For the $0°$ plies (with the x-axis of the laminate in the $0°$ direction) no transformation of axes is necessary:

$$\bar{\mathbf{Q}}_0 = \mathbf{Q} .$$

For the $+60°$ plies, substituting $\theta = 60°$ in the transformation matrix **T** in equation (8.6):

$$\mathbf{T}_{+60} = \begin{bmatrix} 0.25 & 0.75 & 0.4330 \\ 0.75 & 0.25 & -0.4330 \\ -0.8660 & 0.8660 & -0.5 \end{bmatrix} .$$

By equation (8.8):

$$\bar{\mathbf{Q}} = \mathbf{T}^T \mathbf{Q}\, \mathbf{T} .$$

Performing the above matrix multiplication in Excel:

$$\bar{\mathbf{Q}}_{+60} = \begin{bmatrix} 20722 & 27567 & 16320 \\ 27567 & 91144 & 44666 \\ 16320 & 44666 & 30547 \end{bmatrix} \text{ N/mm}^2.$$

For the $-60°$ plies:

$$\bar{\mathbf{Q}}_{-60} = \begin{bmatrix} 20722 & 27567 & -16320 \\ 27567 & 91144 & -44666 \\ -16320 & -44666 & 30547 \end{bmatrix} \text{ N/mm}^2$$

(identical to $\bar{\mathbf{Q}}_{+60}$ except for the sign of the off-diagonal terms in the last row and column).

Summing corresponding terms of the $\bar{\mathbf{Q}}$-matrices for each of the three ply directions, with the appropriate number of plies and ply thickness, according to equation (8.11):

$$A_{11} = 2 \times 4 \times 0.125 \times 20722 + 8 \times 0.125 \times 150905 = 171627 \quad \text{N/mm},$$

$$A_{33} = 2 \times 4 \times 0.125 \times 30547 + 8 \times 0.125 \times 6000 = 36547 \text{ N/mm}.$$

Full A-matrix (from the spreadsheet):

$$\mathbf{A} = \begin{bmatrix} 171628 & 30586 & 0 \\ 30586 & 101205 & 0 \\ 0 & 0 & 36549 \end{bmatrix} \quad \text{N/mm.}$$

Elastic constants, by equations (8.14):

$$A_0 = A_{11}A_{22} - A_{12}^{\;2} = 171628 \times 101205 - (30586)^2 = 16434 \times 10^6 .$$

$$E_x = \frac{1}{t} \cdot \frac{A_0}{A_{22}} = \frac{1}{2.0} \cdot \frac{16434 \times 10^6}{101205} = 81192 \quad \text{N/mm}^2,$$

$$E_y = \frac{1}{t} \cdot \frac{A_0}{A_{11}} = \frac{1}{2.0} \cdot \frac{16434 \times 10^6}{171628} = 47877 \quad \text{N/mm}^2,$$

$$G_{xy} = \frac{A_{33}}{t} = \frac{36549}{2.0} = 18275 \quad \text{N/mm}^2,$$

$$\nu_{xy} = \frac{A_{12}}{A_{22}} = \frac{30586}{101205} = 0.3022, \qquad \nu_{yx} = \frac{A_{12}}{A_{11}} = \frac{30586}{171628} = 0.1782 .$$

The above values agree with the elastic constants given by the spreadsheet.

Exercise 8.2

With thickness $t = 2.0$ mm, the average stress in the laminate is:

$$\sigma_x = \frac{1000}{2.0} = 500 \text{ N/mm}^2.$$

Use elastic constants E_x, E_y, ν_{xy} from Exercise 8.1 to calculate the strains in laminate axes:

$$\varepsilon_x = \frac{500}{81192} = 0.006158\,,$$
$$\varepsilon_y = -0.3022 \times 0.006158 = -0.001861\,,$$
$$\gamma_{xy} = 0\,.$$

The above strains are also the strains in the $0°$ plies.

The stresses in the $0°$ plies, by equation (8.2), with the ply data in Table 8.1 and $\nu_{12} = 0.02$, $\mu = 0.994$ from Example 8.1 are:

$$\sigma_1 = \frac{150000}{0.994} \times 0.006158 - \frac{0.02 \times 150000}{0.994} \times 0.001861 = 923.7 \text{ N/mm}^2,$$
$$\sigma_2 = -\frac{10000}{0.994} \times 0.001861 + \frac{0.3 \times 10000}{0.994} \times 0.006158 = 0\,,$$
$$\tau_{12} = 0\,.$$

For the $+60°$ plies use the transformation matrix calculated in Exercise 8.1:

$$\mathbf{T}_{+60} = \begin{bmatrix} 0.25 & 0.75 & 0.4330 \\ 0.75 & 0.25 & -0.4330 \\ -0.8660 & 0.8660 & -0.5 \end{bmatrix}$$

(note the change of sign for rotation from laminate axes to ply axes).

The strains in ply axes are given by the first of equations (8.5):

$$\boldsymbol{\varepsilon}_{12} = \mathbf{T}\boldsymbol{\varepsilon}_{xy}\,.$$

Performing the above matrix multiplication in Excel:

$$\varepsilon_1 = 0.0001438\,,$$
$$\varepsilon_2 = 0.004153\,,$$
$$\gamma_{12} = 0.006945\,.$$

Stresses in the $\pm 60°$ plies are:

$$\sigma_1 = \frac{150000}{0.994} \times 0.0001438 + \frac{0.02 \times 150000}{0.994} \times 0.004153 = 34.2 \ \ \text{N/mm}^2,$$

$$\sigma_2 = -\frac{10000}{0.994} \times 0.004153 + \frac{0.3 \times 10000}{0.994} \times 0.0001438 = 42.2 \ \ \text{N/mm}^2$$

$$\tau_{12} = 6000 \times 0.006945 = 41.7 \ \ \text{N/mm}^2.$$

The above stresses agree with the stresses given by the spreadsheet.

Exercise 8.3

Value of the Tsai-Hill criterion for each ply direction, with the stresses calculated in Exercise 8.2 and material strength data in Table 8.1

$\underline{0° \text{ plies}}$:

$$\left(\frac{923.7}{2000}\right)^2 = 0.2133$$

$\underline{\pm 60° \text{ plies}}$:

$$\left(\frac{34.2}{2000}\right)^2 - \frac{34.2 \times 42.2}{(2000)^2} + \left(\frac{42.2}{80}\right)^2 + \left(\frac{41.7}{160}\right)^2$$

$$= 0.0003 + 0.0004 + 0.2783 + 0.0679 = 0.3469 .$$

The above values agree with those given by the spreadsheet.

The largest contribution to the Tsai-Hill criterion is due to the transverse stress σ_2 in the $\pm 60°$ plies.

First-ply failure is predicted to occur at load:

$$\frac{1000}{\sqrt{0.3469}} = 1698 \text{ N.}$$

Exercise 8.6

With the ply strength data in Table 8.1 the required thickness of the plies is:

<u>0° plies:</u>

$$t_0 = \frac{N_x}{X_t} = \frac{2000}{2000} = 1.0 \text{ mm} \quad \rightarrow \text{ 8 plies}$$

or

$$t_0 = 0 \text{ mm}.$$

<u>±45° plies:</u>

$$t_{+45} = t_{-45} = \frac{N_{xy}}{X_c} = \frac{1000}{1200} = 0.833 \text{ mm} \quad \rightarrow \mathbf{7} \text{ plies}$$

or

$$t_{+45} = t_{-45} = \frac{N_{xy}}{X_c} = \frac{500}{1200} = 0.417 \text{ mm} \quad \rightarrow \text{ 4 plies.}$$

This is increased to 8 plies at $\pm 45°$ for a symmetric laminate.

<u>90° plies:</u>

$$t_{90} = \frac{N_y}{X_t} = \frac{500}{2000} = 0.25 \text{ mm} \quad \rightarrow \text{ 2 plies}$$

or

$$t_{90} = \frac{N_y}{X_t} = \frac{1500}{2000} = 0.75 \text{ mm} \quad \rightarrow \text{ 6 plies.}$$

Taking the larger number of plies in each group, the total number of plies is 30, with a laminate thickness of 3.75 mm.

This can be interpreted as the following lay-up:

$$[\pm 45_2 / 0_2 / \pm 45 / 90_3 / \pm 45 / 0_2]_S \text{ ,}$$

conforming to the practical restrictions on lay-up in Table 8.2.

Exercise 8.7

θ degree	number of plies	largest Tsai-Hill value	dominant failure mode	action
0 +45 -45 90	8 [(1)] 8 8 6	1.408 in the $\pm 45^\circ$ plies in case 1	transverse $\sigma_2 / Y_t = 1.11$	increase the $\pm 45^\circ$ plies to 10 [(2)]
0 +45 -45 90	8 10 10 6	1.036 in the $\pm 45^\circ$ plies in case 1	transverse $\sigma_2 / Y_t = 0.95$	increase the $\pm 45^\circ$ plies to 12
0 +45 -45 90	8 12 12 6	0.821 in the 90° plies in case 1	transverse $\sigma_2 / Y_t = 0.84$	see below [(3)]
0 +45 -45 90	7 12 12 6	0.939 in the 90° plies in case 1		

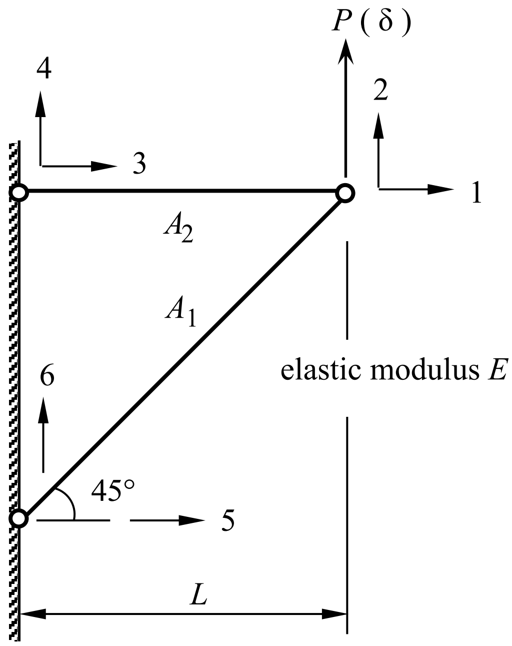
[(1)] Initial number of plies from Exercise 8.6.

[(2)] An even number of plies is required for a symmetric laminate, except that an odd number of either 0° or 90° plies is permissible if located on the middle-plane.

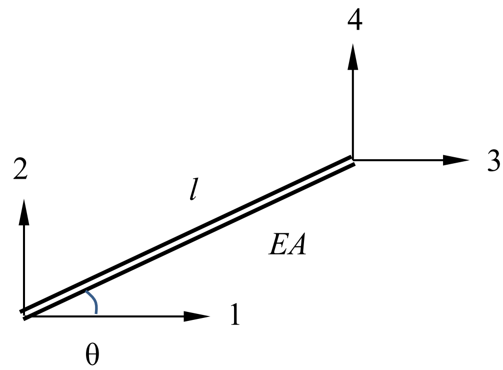
[(3)] At this stage the smallest Tsai-Hill value (= 0.681) is in the 0° plies. By reducing the number of 0° plies from 8 to 7 the largest Tsai-Hill value is increased to 0.939.

The total number of plies is now 37, with a laminate thickness of 4.625 mm.

Exercise 9.1



Stiffness matrix for a bar element:



$$k = \frac{EA}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \qquad \begin{array}{l} c = \cos\theta \\ s = \sin\theta \end{array}$$

Element 1 (area A_1) in the degrees of freedom of the complete structure:

$$c = s = \frac{1}{\sqrt{2}}, \qquad l = \sqrt{2} \cdot L$$

$$k = \frac{E}{L}\begin{bmatrix} \frac{A_1}{2\sqrt{2}} & \frac{A_1}{2\sqrt{2}} & 0 & 0 & -\frac{A_1}{2\sqrt{2}} & -\frac{A_1}{2\sqrt{2}} \\ \frac{A_1}{2\sqrt{2}} & \frac{A_1}{2\sqrt{2}} & 0 & 0 & -\frac{A_1}{2\sqrt{2}} & -\frac{A_1}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{A_1}{2\sqrt{2}} & -\frac{A_1}{2\sqrt{2}} & 0 & 0 & \frac{A_1}{2\sqrt{2}} & \frac{A_1}{2\sqrt{2}} \\ -\frac{A_1}{2\sqrt{2}} & -\frac{A_1}{2\sqrt{2}} & 0 & 0 & \frac{A_1}{2\sqrt{2}} & \frac{A_1}{2\sqrt{2}} \end{bmatrix}$$

Element 2 (area A_2) in the degrees of freedom of the complete structure:

$$c = 1, \qquad s = 0, \qquad l = L$$

$$k = \frac{E}{L}\begin{bmatrix} A_2 & 0 & -A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A_2 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Assembled stiffness matrix:

$$K = \frac{E}{L}\begin{bmatrix} \left(\frac{A_1}{2\sqrt{2}} + A_2\right) & \frac{A_1}{2\sqrt{2}} & -A_2 & 0 & -\frac{A_1}{2\sqrt{2}} & -\frac{A_1}{2\sqrt{2}} \\ \frac{A_1}{2\sqrt{2}} & \frac{A_1}{2\sqrt{2}} & 0 & 0 & -\frac{A_1}{2\sqrt{2}} & -\frac{A_1}{2\sqrt{2}} \\ -A_2 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{A_1}{2\sqrt{2}} & -\frac{A_1}{2\sqrt{2}} & 0 & 0 & \frac{A_1}{2\sqrt{2}} & \frac{A_1}{2\sqrt{2}} \\ -\frac{A_1}{2\sqrt{2}} & -\frac{A_1}{2\sqrt{2}} & 0 & 0 & \frac{A_1}{2\sqrt{2}} & \frac{A_1}{2\sqrt{2}} \end{bmatrix}$$

Deleting rows and columns 3, 4, 5, 6 to constrain displacements at the two supports:

$$K = \frac{E}{L}\begin{bmatrix} \frac{A_1}{2\sqrt{2}} + A_2 & \frac{A_1}{2\sqrt{2}} \\ \frac{A_1}{2\sqrt{2}} & \frac{A_1}{2\sqrt{2}} \end{bmatrix}$$

Exercise 9.2

By equation (9.4) :

$$\frac{\partial g}{\partial A_2} = -\,\mathbf{h}_j^T \mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial A_2} \mathbf{u} \,.$$

In the above formula, $\mathbf{h}^T$, $\mathbf{K}^{-1}$, $\mathbf{u}$ are unchanged from Example 9.1.

To calculate $\dfrac{\partial \mathbf{K}}{\partial A_2}$:

$$K = \frac{E}{L}\begin{bmatrix} \dfrac{A_1}{2\sqrt{2}} + A_2 & \dfrac{A_1}{2\sqrt{2}} \\ \dfrac{A_1}{2\sqrt{2}} & \dfrac{A_1}{2\sqrt{2}} \end{bmatrix}$$

Differentiating:

$$\frac{\partial \mathbf{K}}{\partial A_2} = \frac{E}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Calculate the sensitivity for $A_1 = A_2 = A$ (as in Example 9.1).

Substituting for $\mathbf{h}^T$, $\mathbf{K}^{-1}$, $\dfrac{\partial \mathbf{K}}{\partial A_2}$, $\mathbf{u}$ in the sensitivity formula above:

$$\frac{\partial g}{\partial A_2} = -\begin{bmatrix} 0 & -1 \end{bmatrix} \cdot \frac{L}{EA}\begin{bmatrix} 1 & -1 \\ -1 & 3.828 \end{bmatrix} \cdot \frac{E}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \frac{PL}{EA}\begin{bmatrix} -1 \\ 3.828 \end{bmatrix}$$

Performing the matrix multiplication:

$$\frac{\partial g}{\partial A_2} = \frac{PL}{EA^2}$$

Analytical formula, from Example 9.1:

$$\delta = \frac{PL}{E}\left(\frac{2\sqrt{2}}{A_1} + \frac{1}{A_2}\right)$$

Differentiating and putting $A_2 = A$:

$$\frac{\partial g}{\partial A_2} = -\frac{\partial \delta}{\partial A_2} = \frac{PL}{EA^2} \quad \text{(as above)}$$